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Brief paper

Performance limits in sensor localization[☆]

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ABSTRACT

In this paper, we study the Cramér–Rao Lower Bound (CRLB) in single-hop sensor localization using measurements derived from received signal strength (RSS), time of arrival (TOA) and bearing, respectively, from a novel perspective. Differently from the existing work, we use a statistical sensor–anchor geometry modeling method, with the result that the trace of the associated CRLB matrix, as a scalar metric for performance limits of sensor localization, becomes a random variable. Given a probability measure for the sensor–anchor geometry, the statistical properties of the metric are analyzed to demonstrate properties of sensor localization. Using the Central Limit Theorems for *U*-statistics, we show that as the number of anchors increases, the metric is asymptotically normal in the RSS/bearing case, and converges to a random variable which is an affine transformation of a chi-square random variable of degree 2 in the TOA case. We provide formulas quantitatively describing the relationship among the mean and standard deviation of the metric, the number of the anchors, the parameters of communication channels, the noise statistics in measurements and the spatial distribution of the anchors. These formulas, though asymptotic in the number of the anchors, in many cases turn out to be remarkably accurate in predicting performance limits, even if the number is small. Simulations are carried out to confirm our results.

1. Introduction

Wireless sensor networks have a wide range of military and civilian applications nowadays, in which location information plays a vital role for it is useful to report the geographic origin of events, to assist in target tracking, to achieve geographically aware routing, to manage sensor networks, and so on (Akyildiz, Su, Sankarasubramaniam, & Cayirci, 2002). A sensor network generally consists of two types of nodes: anchors and sensors. Anchor locations are known a priori through GPS or manual configuration, while sensor locations are not known and need to be determined through the procedures of sensor localization; see e.g. Cao, Anderson, and Morse (2006).

Single-hop sensor localization means that each sensor can determine its own location using direct measurements from nearby anchors, and can be found in many practical localization scenarios, such as source localization and target tracking. Effectively, it is used too in simultaneous localization and mapping (SLAM) (Dissanayake, Newman, Clark, Durrant-Whyte, & Csorba, 2001), where a mobile robot equipped with a GPS receiver moves in a 2-dimensional (2-D) environment, measures relative location information to various objects, and then determines the locations of these objects; herein, the positions where the robot makes measurements can be abstracted as *anchors*, such that the localization procedure is single-hop. In Pathirana, Bulusu, Savkin, and Jha (2005), a mobile anchor(s) is used to assist in sensor localization by providing relative location measurements to sensors at multiple positions, which is evidently single-hop. Therefore, it is especially meaningful to study single-hop sensor localization.

Apart from localization algorithms, determining the performance limit of sensor localization, namely the lowest achievable error bound for location estimates, also attracts much attention.

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On the one hand, it provides a measure of theoretically optimal performance regardless of sensor localization algorithms; on the other hand, it reflects fundamentals of sensor localization. Since the Cramér–Rao lower bound (CRLB) establishes a lower bound on the variance for any unbiased estimator, it has been widely used in the performance limit analysis of sensor localization; see e.g. Chang and Sahai (2004), and Patwari, Hero, Perkins, Correal, and O'Dea (2003).

For single-hop sensor localization in a 2-D plane, the CRLB is a 2×2 matrix and dependent on multiple factors, including measuring techniques, noise statistics of measurements, and sensor–anchor geometries (i.e. node coordinates). Since the trace of the CRLB matrix is the minimum *mean square estimation error (MSE)*, it is often used as a *scalar metric* for the performance limit (Chang & Sahai, 2004; Patwari et al., 2003). Provided that the measuring technique and the noise statistics of measurements are both known, the scalar metric can be regarded as a function of the sensor–anchor geometry. A key problem arising is that of minimizing the scalar metric, equivalent to identifying optimal sensor–anchor geometries for sensor localization, and has been widely studied (Bishop, Fidan, Anderson, Dogcancay, & Pathirana, 2010; Martinez & Bullo, 2006). The metric can also give valuable qualitative information. It can be very large, implying that a localization problem is badly conditioned (e.g. the anchors are nearly collinear with the sensor) and localization algorithms almost fail. Evidently, we should avoid the situations where the scalar metric takes large values. In short, the CRLB provides much useful information regarding sensor localization.

The conventional CRLB studies assume a deterministic sensor–anchor geometry. But, the sensor–anchor geometry is usually unknown prior to system deployment, so that it is difficult to evaluate the localization performance. Yet, a probability measure for the sensor–anchor geometry might be available. It is indeed natural to model the senor–anchor geometry by assuming a random and uniform distribution for the anchors' positions, and consequently, the scalar metric itself becomes random and offers a broad, statistical view on the localization performance, in contrast to one deterministic quantity for a given sensor–anchor geometry. For instance, if the scalar metric hardly ever takes large values, there is less reason to worry about the sensor–anchor geometry; otherwise, one must impose proper control on it. Additionally, the mean of the scalar metric further establishes a lower limit on the performance of single-hop sensor localization given a fixed number of anchors with undetermined locations; in the situations where sensor–anchor geometries are unknown, e.g. prior to system deployment, this performance limit is certainly useful.

The novel statistical sensor–anchor geometry modeling method not only provides insights into single-hop sensor localization and in turn guides us in the design and deployment of wireless sensor networks, but also as a prototype paves the way for dealing with more complicated scenarios of sensor localization. In a mobile environment, as may arise with ad-hoc networks, SLAM, mobile anchors assisting in sensor localization and so on, it is trivial to concentrate on localization performance in one particular time instant, whereas it is evidently more attractive to acquire the knowledge about the average localization performance over a period of time and/or in a wide region; hopefully, these challenges can be addressed by the statistical modeling method. In summary, statistical sensor–anchor geometry modeling is a powerful method for investigating the performance limit of sensor localization. To the best of our knowledge, this method has never been considered.

In this paper, we consider single-hop sensor localization based on received signal strength (RSS), time of arrival (TOA) and bearing, respectively, and show that the scalar metric in each

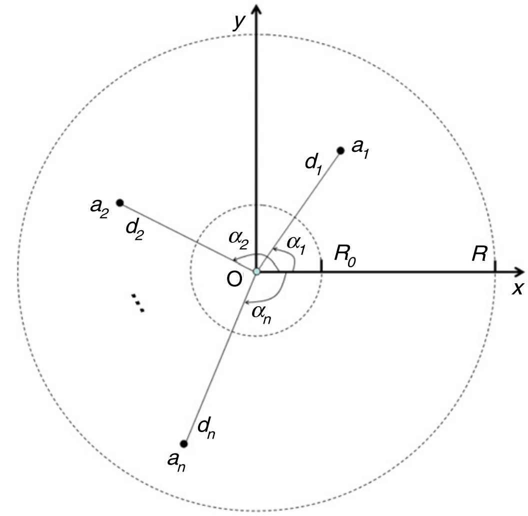

Fig. 1. Single-hop sensor localization.

case is essentially a function of U-statistics (see Section 2.2 for further details). Based on the theory of U-statistics, we make the following contributions: (i) it is proved that as the number of the anchors increases, the scalar metric is asymptotically *normal* in the RSS/bearing case and converges to a random variable (RV) which is an affine transformation of a *chi-square* RV of degree 2 in the TOA case; (ii) the convergence rate in the RSS/bearing case is shown to be as fast as $O(n^{-\frac{1}{2}})$, where n is the number of the anchors; (iii) the *asymptotic* formulas for the mean and standard deviation of the scalar metric are derived in both cases; (iv) last but not least, these formulas are analyzed to demonstrate some properties of sensor localization. All conclusions are confirmed by extensive simulations.

The remainder of this paper is organized as follows. The next section formulates the problem of single-hop RSS-based sensor localization and presents the main results. Section 3 extends our study into bearing-only localization and TOA-based localization. Finally, we conclude this paper and shed light on future work in Section 4.

Throughout this paper, we use the following mathematical notation: $(\cdot)^T$ denotes the transpose of a matrix or a vector; $Tr(\cdot)$ denotes the trace of a square matrix; $Pr\{\cdot\}$ denotes the probability of an event; $E_x(\cdot)$ and $Std_x(\cdot)$ denote the statistical expectation and standard deviation with respect to the subscripted RV x.

2. Results in RSS-based localization

2.1. Problem formulation

In a 2-D plane, consider a single sensor (or source, target) located at the origin and n measurements made from this sensor to n anchors at known locations $\{\mathbf{a}_i, i = 1, \ldots, n\}$, as illustrated in Fig. 1. Here we shall consider RSS measurements, but later distance (TOA) or angle measurements as alternatives. The true distance from the sensor to the i-th anchor is denoted by d_i; the true angle subtended by $\mathbf{a}_i$ and the positive x-axis is denoted by α_i. The task of single-hop sensor localization is to find an estimate of the sensor location using $\{\mathbf{a}_i, i = 1, \ldots, n\}$ and the associated distance or angle measurements.

Denote by $\{P_i, i = 1, \ldots, n\}$ the RSS measurements transmitted by the single sensor (or by the n anchors with the same transmission power) at the n anchors (or at the single sensor). Then, we assume the following.

Assumption 1. The RSS measurements $\{P_i, i = 1, \ldots, n\}$ satisfy the log-normal (shadowing) model and are statistically independent.

Assumption 1 is commonly made in studies on RSS-based sensor localization (e.g. Patwari et al. (2003), and So and Lin (2011)). This assumption is two-fold: the first aspect postulates the log-normal model which has been widely verified (e.g. Patwari et al. (2003)) and the second aspect postulates the independence between RSS measurements which is verified in Kaemarungsi and Krishnamurthy (2012). As pointed out in Patwari, Wang, and O'Dea (2002), given two spatially close anchors, the statistical independence between the associated two RSS measurements may be compromised, because an obstruction such as a wall, furniture, tree, or building may cause similar shadowing, and hence, the second aspect would not properly hold in this case.

With the RSS measurements, one can obtain distance measurements (Chitte, Dasgupta, & Ding, 2009) to estimate the sensor location. Regarding the location estimation problem, the Fisher information matrix (FIM) F_{RSS} is formulated as follows:

$$F_{\text{RSS}} = b \begin{pmatrix} \sum_{i=1}^{n} \frac{\cos^2 \alpha_i}{d_i^2} & \sum_{i=1}^{n} \frac{\cos \alpha_i \sin \alpha_i}{d_i^2} \\ \sum_{i=1}^{n} \frac{\cos \alpha_i \sin \alpha_i}{d_i^2} & \sum_{i=1}^{n} \frac{\sin^2 \alpha_i}{d_i^2} \end{pmatrix}, \tag{1}$$

where $b = \left(\frac{10\alpha}{\sigma_{dB} \ln 10}\right)^2$, α is the path-loss exponent and σ_{dB} is the standard deviation of the shadowing effect in the log-normal model. A detailed derivation can be found in Patwari et al. (2003).

Let $(\hat{x}, \hat{y})$ be the unbiased sensor location estimate and C_{RSS} be the CRLB on the covariance of $(\hat{x}, \hat{y})$. If F_{RSS} is non-singular, C_{RSS} equals F_{RSS}^{-1} and satisfies

$$Tr(C_{\text{RSS}}) \leq E_{\boldsymbol{\upsilon}}(\hat{x}^2 + \hat{y}^2), \tag{2}$$

where $\boldsymbol{\upsilon} = \{P_i, i = 1, \ldots, n\}$. As such, $Tr(C_{\text{RSS}})$ is the *scalar metric* for the performance limit with the expression:

$$Tr(C_{\text{RSS}}) = \frac{1}{b} \left(\frac{\sum_{i=1}^{n} \frac{1}{d_i^2}}{\sum_{1 \leq i < j \leq n} \frac{\sin^2(\alpha_i - \alpha_j)}{d_i^2 d_j^2}} \right). \tag{3}$$

Next, we define the random sensor–anchor geometry model by assuming the following.

Assumption 2. The n anchors are randomly and uniformly distributed inside an annulus centered at the sensor and defined by radii R_0 and R $(R > R_0 > 0)$.

Here, R is the upper bound on practical distances used in the wireless communication system and is normally restricted by the physical factors determining path-loss attenuations; R_0 is the reference distance defined in the log-normal model and is virtually the lower bound on distances that can be measured by using the log-normal model. For example, based on measurement studies, the reference distance R_0 for practical systems using low-gain antennas in the 1–2 GHz region is typically chosen to be 1 m in indoor environments and 0.1 km or 1 km in outdoor environments (Rappaport, 2001).

By Assumption 2, each possible sensor–anchor geometry is equi-probable, in the sense that the sensor–anchor geometry follows a "uniform" distribution. As a result, $\boldsymbol{\omega} = \{d_i, \alpha_i, i = 1, \ldots, n\}$ are mutually independent RVs and the scalar metric $Tr(C_{\text{RSS}})$ becomes random.

2.2. Results

In Hoeffding (1948), U-statistics (which crucially obey a central limit theorem) are defined as follows.

Definition 1. Let $\{X_i, i = 1, \ldots, n\}$ be i.i.d. p-dimensional random vectors. Let $h(x_1, \ldots, x_r)$ be a Borel function on $\mathbb{R}^{r \times p}$ for a given positive integer r $(\leq n)$ and be symmetric in its arguments. A U-statistic U_n is defined by

$$U_n = \frac{r!(n-r)!}{n!} \sum_{1 \leq i_1 < \cdots < i_r \leq n} h(X_{i_1}, \ldots, X_{i_r}) \tag{4}$$

and $h(x_1, \ldots, x_r)$ is called the kernel of U_n.

Obviously, $Tr(C_{\text{RSS}})$ involves the ratio of two U-statistics, which inspires us to study $Tr(C_{\text{RSS}})$ through an asymptotic analysis based on the theory of U-statistics. At first, we obtain the following lemma for processing the ratio of two U-statistics (Due to the page limit, we cannot provide the proofs for the following lemma and all the other theorems in this paper; all these proofs can be found in http://arxiv.org/abs/1109.2984v1).

Lemma 1. *Given two sequences of i.i.d. RVs with bounded values: $\{X_i^{(1)}, i = 1, \ldots, n\}$ and $\{X_i^{(2)}, i = 1, \ldots, n\}$, which are mutually independent, define vectors $X_i = [X_i^{(1)} \quad X_i^{(2)}]^T$ $(i = 1, \ldots, n)$ and two U-statistics*

$$T_n = \frac{1}{n} \sum_{i=1}^{n} X_i^{(1)}, \tag{5}$$

$$S_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} X_i^{(1)} X_j^{(1)} \sin^2(X_i^{(2)} - X_j^{(2)}). \tag{6}$$

Then, (a)

$$\frac{T_n}{S_n} = \frac{1}{m_1 m_2} + \frac{2\sigma_1^2}{n m_1^3 m_2} + M_n + R_n \tag{7}$$

where $m_1 = E(X_1^{(1)})$, $\sigma_1 = Std(X_1^{(1)})$, $m_2 = E(\sin^2(X_1^{(2)} - X_2^{(2)}))$, and

$$M_n = \frac{2}{n} \sum_{i=1}^{n} g_1(X_i) + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} g_2(X_i, X_j), \tag{8}$$

$$g_1(X_i) = \frac{m_1 - X_i^{(1)}}{2m_1^2 m_2}, \tag{9}$$

$$g_2(X_i, X_j) = \frac{1}{m_1 m_2} - \frac{X_i^{(1)} + X_j^{(1)}}{m_1^2 m_2} + \frac{2X_i^{(1)} X_j^{(1)}}{m_1^3 m_2} - \frac{X_i^{(1)} X_j^{(1)} \sin^2(X_i^{(2)} - X_j^{(2)})}{m_1^3 m_2^2}; \tag{10}$$

(b) *for any $\varepsilon > 0$, as $n \to \infty$, the remainder term R_n satisfies*

$$Pr\{|nR_n| \geq \varepsilon\} = O(n^{-1}), \tag{11}$$

$$Pr\{|n(\ln n)R_n| \geq \varepsilon\} = o(1). \tag{12}$$

Lemma 1 allows us to expand $Tr(C_{\text{RSS}})$. By letting $X_i^{(1)} = \frac{1}{d_i^2}$ and $X_i^{(2)} = \alpha_i$, we can derive $m_2 = 0.5$, and

$$m_1 = 2\left(\frac{\ln \frac{R}{R_0}}{R^2 - R_0^2}\right), \tag{13}$$

$$\sigma_1 = \sqrt{\frac{1}{R_0^2 R^2} - \left(\frac{2\ln \frac{R}{R_0}}{R^2 - R_0^2}\right)^2}. \tag{14}$$

Then, one of our main results is further summarized as follows.

Theorem 2. *Define a sequence of RVs*

$$W_n = \left(\frac{\sqrt{n}(n-1)bm_1^2}{4\sigma_1}\right) Tr(C_{RSS}) - \frac{\sqrt{n}m_1}{\sigma_1} - \frac{2\sigma_1}{\sqrt{n}m_1} \tag{15}$$

where b and $Tr(C_{RSS})$ are defined in Section 2.1, and m_1 and σ_1 are defined by (13) and (14). Then, under Assumptions 1 and 2, W_n converges in distribution to a standard normal RV as $n \to \infty$.

Remark 3. In view of the affine relationship between W_n and $Tr(C_{RSS})$, $Tr(C_{RSS})$ is asymptotically normal. Given a sufficiently large n, the distribution of $Tr(C_{RSS})$ can be approximated by the following normal distribution

$$\mathcal{N}\left(\frac{4 + \frac{8\sigma_1^2}{nm_1^2}}{(n-1)bm_1}, \left(\frac{4\sigma_1}{\sqrt{n}(n-1)bm_1^2}\right)^2\right), \tag{16}$$

and furthermore, the moments of $Tr(C_{RSS})$ can be approximated as follows:

$$E_\omega(Tr(C_{RSS})) \approx \frac{4 + \frac{8\sigma_1^2}{nm_1^2}}{(n-1)bm_1}, \tag{17}$$

$$Std_\omega(Tr(C_{RSS})) \approx \frac{4\sigma_1}{\sqrt{n}(n-1)bm_1^2}. \tag{18}$$

With (16)–(18), we can quantitatively analyze the localization performance, e.g. (a) computing the probability that $Tr(C_{RSS})$ is below a given threshold for a known value of n, (b) determining a threshold such that $Tr(C_{RSS})$ is below the threshold with a certain confidence level, (c) determining the minimum n such that $Tr(C_{RSS})$ is below a given threshold with a certain confidence level, and (d) computing the spatial average of the localization performance given a known value of n, which is undoubtedly helpful for the design and deployment of wireless sensor networks.

A natural question arises as to how large n should be to obtain a good approximation; this gives rise to the convergence rate study. The following theorem describes the convergence rate of W_n.

Theorem 4. *Under Assumptions 1 and 2, as $n \to \infty$,*

$$\sup_x |F_n(x) - \Phi(x)| \le \left|\left(\frac{\nu_3 + \frac{2\sigma_1^4}{m_1}}{6\sigma_1^3}\right)\frac{(x^2-1)e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right| n^{-\frac{1}{2}} + O(n^{-1}) \tag{19}$$

where $F_n(x)$ and $\Phi(x)$ are the distribution functions (DFs) of W_n and the standard normal RV, $\nu_3 = E_\omega((1/d_1^2 - m_1)^3)$, and the other notation are the same as in Theorem 2.

Remark 5. It follows that as $n \to \infty$, the density of W_n converges to standard normality with a relatively fast rate, i.e. $O(n^{-\frac{1}{2}})$. Additionally, it can be verified that the coefficient associated with

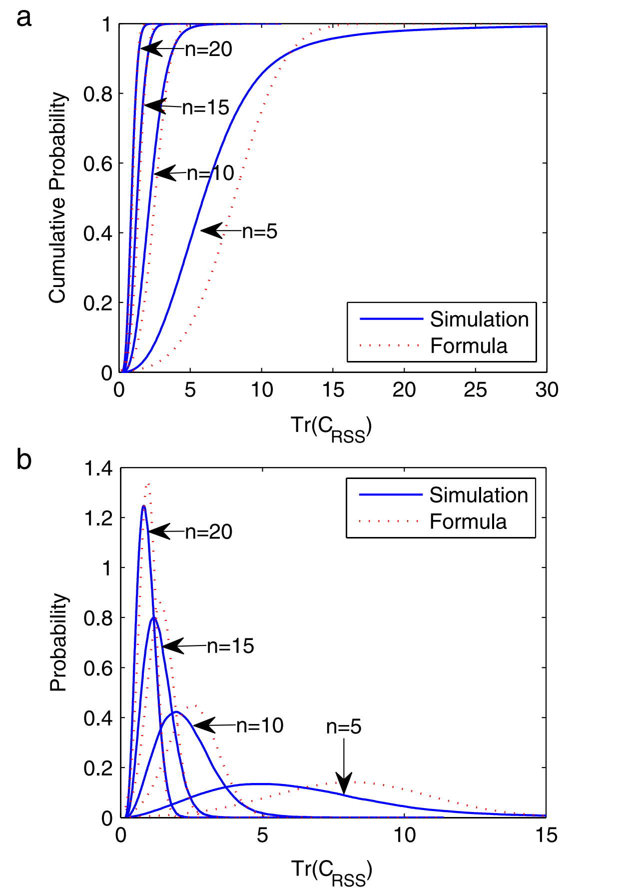

Fig. 2. The DFs and PDFs of $Tr(C_{RSS})$ with $\alpha = 2.3$, $\sigma_{dB} = 3.92$, $R_0 = 1$ m and $R = 10$ m.

$n^{-\frac{1}{2}}$ in (19) is a function of $\frac{R}{R_0}$; that is to say, the convergence rate is not determined by the individual values of R_0 and R, but only by $\frac{R}{R_0}$.

To conduct simulations to verify Theorem 2, the parameters describing a typical wireless channel, i.e. α, σ_{dB} and R_0, are set as 2.3, 3.92 and 1 m, respectively, as measured in Patwari et al. (2003). The DFs and probability density functions (PDFs) of $Tr(C_{RSS})$ from both simulations (with the legend "Simulation") and (16) (with the legend "Formula") are plotted in Fig. 2. The curves from simulations are obtained by generating 100,000 single-hop sensor localization scenarios, evaluating the CRLB in each scenario, and then computing the DFs and PDFs using the Matlab routine "ksdensity".

The gradually diminishing discrepancy with increasing n between the pair of curves in Fig. 2 is consistent with and in turn confirms Theorem 2. In addition, with n increasing, the value range of $Tr(C_{RSS})$ becomes narrower (or in other words, the dispersion of the DF of $Tr(C_{RSS})$ reduces), implying that the sensitivity of localization performance to sensor–anchor geometries reduces; hence, one should be careful about sensor–anchor geometries when n is small, but has less reason to worry about them when n is large.

Remark 6. As illustrated in Fig. 2, when $n \ge 10$, (16) achieves good performance and meanwhile (17) and (18) are applicable, which enables us to analytically study the properties of sensor localization. (Since $E_\omega(Tr(C_{RSS}))$ and $Std_\omega(Tr(C_{RSS}))$ in (17) and (18) normalized by R^2 (or R_0^2) are dependent on $\frac{R}{R_0}$, we simplify the discussion involving both R_0 and R by letting $R_0 = 1$ m and only concentrating on R.)

(1) From (17), $E_\omega(Tr(C_{\mathrm{RSS}}))$ is inversely proportional to n, and thus a *critical* value n^* depending on R_0, R, σ_{dB} and α can be determined, such that having more anchors than n^* contributes little to improving the quality of sensor localization.
(2) By (17) and (18), $E_\omega(Tr(C_{\mathrm{RSS}}))$ and $Std_\omega(Tr(C_{\mathrm{RSS}}))$ both decrease with R decreasing; the reason is that long distance measurements from RSS suffer greater errors, and thus produce worse localization performance. Therefore, distance measurements from a sensor are better made at locations as close to the sensor as possible.
(3) It follows from (17) that using more distance measurements spread over a wide range is not necessarily better than using fewer distance measurements but spread in a narrow range in terms of $E_\omega(Tr(C_{\mathrm{RSS}}))$. Hence, tradeoff should be made between the number of the anchors (i.e. n) and their spreading determined by R_0, R.
(4) Though the impacts of increasing n and R are discussed separately, the variables are correlated in some situations, and so the impacts are related. This is because increasing the transmission power enlarges the node communication coverage, and both n and R tend to rise. However, a positive impact arises for $Tr(C_{\mathrm{RSS}})$ as its mean will definitely decrease according to Patwari et al. (2003).
(5) Since the coefficient of variation of a RV, defined to be the ratio of its standard deviation to its mean, is a normalized measure of dispersion of its distribution, such a coefficient is indeed a metric for the sensitivity of localization performance to sensor–anchor geometries. By (17) and (18), the coefficient associated with $Tr(C_{\mathrm{RSS}})$ has the order of $O(n^{-\frac{1}{2}})$, implying that the sensitivity dies out with n going to infinity.

3. Expansion in bearing-only localization and TOA-based localization

In this section, we expand our study into bearing-only localization and TOA-based localization.

3.1. Results in bearing-only localization

In bearing-only localization, bearing measurements associated with one sensor and at least two anchors noncollinear with the sensor are required to determine the sensor location. We still consider the sensor and n (≥ 2) anchors in Fig. 1, but $\{\alpha_i, i = 1, \ldots, n\}$ are assumed to be measured as bearings, because this set of measurements is equivalent to the set of real bearing measurements as far as our study is concerned. Henceforth, we make the following assumption as is commonly used in the studies on bearing-only localization (Dogancay, 2005; Gadre, Roan, & Stilwell, 2008) as well as is also justified in Gadre et al. (2008).

Assumption 3. The bearing measurements are statistically independent and Gaussian with means $\{\alpha_i, i = 1, \ldots, n\}$ and the same variance σ_α^2.

To model a random sensor–anchor geometry in bearing-only localization, we make the same assumption as Assumption 2. Here, R_0 is not the reference distance any more, but the lower bound on practical distances from the sensor to anchors. Define F_B to be the FIM in bearing-only localization, and from Bishop et al. (2010), we have

$$F_B = \frac{1}{\sigma_\alpha^2}\begin{pmatrix} \sum_{i=1}^{n} \frac{\cos^2\alpha_i}{d_i^2} & \sum_{i=1}^{n} -\frac{\cos\alpha_i \sin\alpha_i}{d_i^2} \\ \sum_{i=1}^{n} -\frac{\cos\alpha_i \sin\alpha_i}{d_i^2} & \sum_{i=1}^{n} \frac{\sin^2\alpha_i}{d_i^2} \end{pmatrix}. \quad (20)$$

Remark 7. Obviously, on replacing $\frac{1}{\sigma_\alpha^2}$ by b, F_B will have the same form as the FIM in the RSS case, i.e. F_{RSS} in (1). Hence, all the conclusions about $Tr(C_{\mathrm{RSS}})$ except those relevant to b are still correct in bearing-only localization under Assumption 3.

3.2. Results in TOA-based localization

By the relation between signal propagation speed in a medium, the travel time of a radio signal (i.e. TOA) is a measure for the distance between the transmitter and the receiver. Denote by $\{T_i, i = 1, \ldots, n\}$ the measured TOA between the sensor and n anchors. Based on the experiments in real environments (Mazomenos, De Jager, Reeve, & White, 2011; Patwari et al., 2003) and as is common in studies on TOA-based localization under line-of-sight conditions (Xu, Ding, & Dasgupta, 2011; Zhu & Ding, 2010), we assume the following.

Assumption 4. $\{T_i, i = 1, \ldots, n\}$ are statistically independent and Gaussian with means $\{\frac{d_i}{c}, i = 1, \ldots, n\}$ (c is the speed of propagation) and the same variance σ_T^2.

Because the TOA measurement model is still valid when the practical distance d_i is 0, we do not impose a lower bound on practical distances as we do in the RSS case, and simply make the following assumption to model a random sensor–anchor geometry.

Assumption 5. The n anchors are randomly and uniformly deployed within a circle of radius R ($R > 0$) centered at the sensor.

Then, $\{d_i, i = 1, \ldots, n\}$ and $\{\alpha_i, i = 1, \ldots, n\}$ are mutually independent and we can obtain the FIM and the scalar metric as follows:

$$F_{\mathrm{TOA}} = \frac{1}{\sigma_T^2 c^2}\begin{pmatrix} \sum_{i=1}^{n} \cos^2\alpha_i & \sum_{i=1}^{n} \cos\alpha_i \sin\alpha_i \\ \sum_{i=1}^{n} \cos\alpha_i \sin\alpha_i & \sum_{i=1}^{n} \sin^2\alpha_i \end{pmatrix}, \quad (21)$$

$$Tr(C_{\mathrm{TOA}}) = \frac{\sigma_T^2 c^2 n}{\sum_{1\leq i<j\leq n} \sin^2(\alpha_i - \alpha_j)} \quad (22)$$

where F_{TOA} and C_{TOA} are the FIM and CRLB respectively in the TOA case.

According to Lemma 1, by letting $X_i^{(1)} = 1$ and $X_i^{(2)} = \alpha_i$ with $i = 1, \ldots, n$, we can derive the following theorem.

Theorem 8. *Define a sequence of RVs*

$$V_n = \left(\frac{n(n-1)}{2\sigma_T^2 c^2}\right) Tr(C_{\mathrm{TOA}}) - 2n + 2 \quad (23)$$

where σ_T is defined in Assumption 4 and $Tr(C_{\mathrm{TOA}})$ is defined by (22). Under Assumptions 4 and 5, V_n converges in distribution to a chi-square RV of degree 2 as $n \to \infty$.

Remark 9. According to Theorem 8, if n is sufficiently large, the PDF of $Tr(C_{\mathrm{TOA}})$ can be approximated by

$$\frac{n(n-1)}{2\sigma_T^2 c^2} f_\chi\left(\frac{n(n-1)}{2\sigma_T^2 c^2}x - 2n + 2\right), \quad (24)$$

where $f_\chi(\cdot)$ is the PDF of the chi-square RV of degree 2, and moreover, the moments of $Tr(C_{\mathrm{TOA}})$ can be approximated as follows,

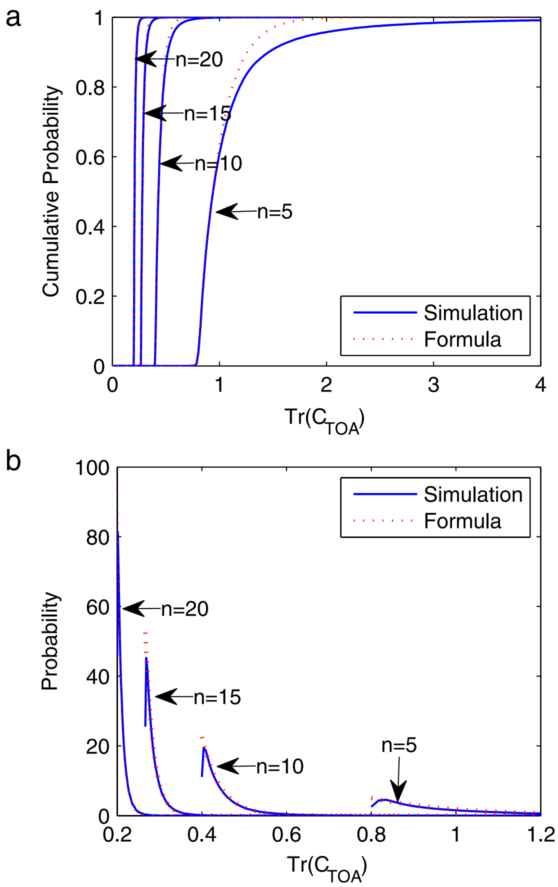

Fig. 3. The DFs and PDFs of $Tr(C_{\mathrm{TOA}})$ with $R = 10$ m and $\sigma_T c = 1$ m.

$$E_\omega(Tr(C_{\mathrm{TOA}})) \approx \frac{4\sigma_T^2 c^2}{n-1}, \tag{25}$$

$$Std_\omega(Tr(C_{\mathrm{TOA}})) \approx \frac{4\sigma_T^2 c^2}{n(n-1)}. \tag{26}$$

We let $\sigma_T c = 1$ m as measured in Patwari et al. (2003), and plot the DFs and PDFs of $Tr(C_{\mathrm{TOA}})$ from both simulations (with the legend "Simulation") and (24) (with the legend "Formula") in Fig. 3. The discrepancy between the pair of curves is not as obvious as in the RSS case, and also vanishes with n increasing, which confirms Theorem 8.

Remark 10. Since (24) is extraordinarily accurate when $n \geq 10$, we analyze the properties of sensor localization using (25) and (26) as in the RSS case. First, it is shown by (25) that $E_\omega(Tr(C_{\mathrm{TOA}}))$ is inversely proportional to n, and a critical value n^* differing from $\sigma_T c$ can be determined, such that having more anchors than n^* does not distinctly improve the quality of sensor localization. Second, the coefficient of variation for $Tr(C_{\mathrm{TOA}})$ has the order of $O(n^{-1})$, implying that TOA-based localization is not as sensitive to sensor–anchor geometries as RSS-based localization.

4. Conclusion and future work

In this paper, we investigated the performance limit of single-hop sensor localization using RSS, TOA or bearing measurements by statistical sensor–anchor geometry modeling. That is, given a probability measure for the sensor–anchor geometry, the scalar metric for the performance limit, i.e. the trace of the associated CRLB matrix, becomes random. We obtained formulas expressing the asymptotic behavior of the scalar metric in terms of distribution, mean and standard deviation. Specifically, as the number of the anchors goes to infinity, the scalar metric in the RSS/bearing case is asymptotically normal and its rate of convergence to normality was also derived; in the TOA case, the scalar metric converges to a RV which is an affine transformation of a chi-square RV of degree 2. Although these formulas are asymptotic in the number of the anchors, extensive simulations show that they are remarkably accurate in predicting the performance limit of sensor localization even if the number of the anchors is fairly small. In addition, we demonstrate some general properties of sensor localization based on the mean and standard deviation of the scalar metric.

Considering the similarities between the models for bearing measurements and angle of arrival (AOA) measurements, we can expand the conclusions in the RSS/bearing case to AOA-based localization. Furthermore, distance measurements in range-only localization are often modeled to be mutually independent and Gaussian (Bishop et al., 2010), which is the same as occurs with TOA measurements, and thus, it is straightforward to expand the conclusions in the TOA case to range-only localization. In future work, we may expand our study into 3-dimensional space and multi-hop sensor localization.

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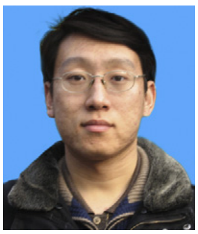

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